

Wigner functions, contact interactions, and matching

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Abstract

Quantum mechanics in phase space (or deformation quantization) appears to fail as an autonomous quantum method when infinite potential walls are present. The stationary physical Wigner functions do not satisfy the normal eigen equations, the \star -eigen equations, unless an ad hoc boundary potential is added [1]. Alternatively, they satisfy a different, higher-order, “ \star -eigen- \star equation”, locally, i.e. away from the walls [2]. Here we show that this substitute equation can be written in a very simple form, even in the presence of an additional, arbitrary, but regular potential. The more general applicability of the \star -eigen- \star equation is then demonstrated. First, using an idea from [3], we extend it to a dynamical equation describing time evolution. We then show that also for general contact interactions, the \star -eigen- \star equation is satisfied locally. Specifically, we treat the most general possible (Robin) boundary conditions at an infinite wall, general one-dimensional point interactions, and a finite potential jump. Finally, we examine a smooth potential, that has simple but different expressions for x positive and negative. We find that the \star -eigen- \star equation is again satisfied locally. It seems, therefore, that the \star -eigen- \star equation is generally relevant to the matching of Wigner functions; it can be solved piece-wise and its solutions then matched.

1 Introduction

Consider the quantum mechanics of a single particle moving in one dimension. Infinite potential walls are handled easily in operator quantum mechanics. The physical wave functions are solutions of the Schrödinger equation away from the walls, and they are simply required to satisfy boundary conditions at the wall locations.

In Wigner-Weyl-Moyal quantum mechanics (or deformation quantization),¹ however, infinite potential walls are surprisingly tricky [1]. Quantum states are described by Wigner functions, the Wigner-Weyl transforms (or symbols) of the corresponding density matrices. Normally, the Wigner functions $\rho(x, p)$ describing stationary states obey the \star -eigen equations (or so-called “star-genvalue” equations)

$$H(x, p) \star \rho(x, p) = \rho(x, p) \star H(x, p) = E \rho(x, p). \quad (1)$$

Here $H(x, p)$ is the classical Hamiltonian, and

$$\star = \exp \left\{ \frac{i\hbar}{2} \left(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x \right) \right\} \quad (2)$$

is the Grönewold-Moyal star product. In the presence of an infinite potential wall, the symbol of the well-known density matrix does not satisfy the \star -eigen equation.

What goes wrong? One possibility is that the symbol of the density matrix is not physical. It was shown that this is not the case, however, in [2] and [6]. In [2], the infinite potential wall was treated as a limit of an exponential (Liouville) potential, for which the \star -eigen equation is solvable [7]. In the $\alpha \rightarrow \infty$ limit of the potential $V_0 e^{2\alpha x}$, its Wigner function approaches the canonical one, the symbol of the usual density matrix. The details were spelled out in [6].

Now, α determines both the height and the “size” (or width) $1/\alpha$ of the potential $V_0 e^{2\alpha x}$. We believe that it is the zero-size $1/\alpha \rightarrow 0$ limit that is the relevant one here. As just described, if the zero-size limit is taken after the $\hbar \rightarrow 0$ limit, the physical result is obtained.

Deformation quantization treats quantum mechanics as an \hbar -deformation of ordinary classical mechanics. It therefore assumes that the usual canonical, classical mechanics is recovered in the $\hbar \rightarrow 0$ limit. Since an infinite potential wall (or any potential with a zero-size feature) is to be understood as the zero-size limit of a smooth potential, we would like to take that limit first, and then \hbar -deform, to get its phase-space quantum mechanics.

¹For elementary introductions, see [4]; for more advanced reviews, consult [5].

The problem arises because the limits $\hbar \rightarrow 0$ and $1/\alpha \rightarrow 0$ do not commute. Put another way, no finite de Broglie wavelength can be considered small relative to the width of a sharp, infinite potential wall [8]. If the zero-size limit ($1/\alpha \rightarrow 0$) is taken first, classical mechanics is not retrieved in the $\hbar \rightarrow 0$ limit.

Certain phenomena make that clear. For example, there are non-Newtonian, or para-classical, reflections present in the $\hbar \rightarrow 0$ limit, that are not described by classical mechanics [8]. To describe them, perhaps some para-classical mechanics² could be deformed, but not normal classical mechanics.³

The situation is even worse, however. Deformation quantization treats quantum mechanics as an \hbar -deformation of a specific treatment of ordinary classical mechanics: the canonical, phase-space formulation. To the best of our knowledge, no such formulation exists for a system with a zero-size potential feature, such as the infinite potential wall.⁴

How can such potentials be treated in pure deformation quantization, i.e., in deformation quantization, considered as an autonomous formulation of quantum mechanics?

Dias and Prata introduced a boundary potential to cure the problem [1]. Consistent with the arguments above, the potential is proportional to \hbar^2 , and so describes non-classical effects. The additional boundary term is ad hoc, however. The original motivation for the work reported in [2] was to derive this term from first principles. This has not yet been achieved.⁵

Instead, in [2] an alternative method of pure deformation quantization was found for systems with such infinite-wall potentials. Most importantly, the result was derived, rather than postulated. For the infinite wall potentials, the Wigner function was shown to satisfy a higher-order \star -equation locally, i.e., away from the walls. The physical Wigner function can be found by solving this new equation, then imposing boundary conditions [10].

If the potential energy consists only of infinite potential walls, the Wigner function was shown to obey the higher-order \star -equation

$$(p^2 - E) \star \rho(x, p) \star (p^2 - E) = 0 \quad (3)$$

²Para-classical reflections have been incorporated into a path-integral formulation in [9].

³These para-classical reflections are present in the operator formulation of quantum mechanics. That formulation does not seem to have difficulty with infinite potential walls, however; it does not describe quantum mechanics as a deformation of classical mechanics.

⁴Of course, such systems can be described as zero-size limits of ones with regular potentials.

⁵See [6], however. It was pointed out there that even in the operator formulation, the free Hamiltonian on the half-line must be extended to a self-adjoint operator by adding a boundary potential at the wall's location. However, the additional boundary potential is not of the precise form proposed by Dias and Prata [1].

locally, i.e. away from those walls. For simplicity, we use units such that $2m = 1$, so that the Hamiltonian away from the wall location is just $H = p^2$. The new “ \star -eigen- \star equation” is therefore, in this case, just

$$(H(x, p) - E) \star \rho(x, p) \star (H(x, p) - E) = 0. \quad (4)$$

In [2], the \star -eigen- \star equation was shown to work for various potentials composed of infinite walls, and wells. It was also generalized to the case when an additional, arbitrary but regular potential is present. The resulting equation had a complicated form – see (6) below. Our first result here is that this complicated expression reduces to the simple equation (4), which is therefore completely general. The demonstration can be found in the next section.

The \star -eigen- \star equation is therefore simpler than was previously thought. This work will also show that it is more generally applicable than was realized.

It should be mentioned that a comparison of the Dias-Prata boundary potential solution [1] and the \star -eigen- \star method [2] was made in [10]. A certain equivalence of the two methods was shown, provided suitable boundary and kinematical conditions are imposed. One difference was also pointed out, however. It seems reasonably straightforward to study dynamics using the Dias-Prata boundary potential – the equation of motion of the Wigner function is the usual evolution equation, but with the boundary potential term included. Time evolution seems problematic using the \star -eigen- \star equation, however. We address this concern in section 3. Borrowing an idea from [3], we show how an equation of motion can be written that reduces to the \star -eigen- \star equation for stationary Wigner functions.

Section 4 is motivated by the arguments above concerning zero-size features of potentials. We examine one-dimensional systems with contact interactions,⁶ i.e., point interactions, or sharp reflecting boundaries. In subsection 4.1, the general, Robin boundary conditions for the half-line are considered, and the general point interaction is treated in subsection 4.2. In all cases, the \star -eigen equation is not satisfied locally, while the \star -eigen- \star equation is, just as for the infinite potential wall. Perhaps most significantly, the *finite* potential wall demonstrates the same behaviour, as shown in section 4.3.

In section 5 we consider a simple potential that has no zero-size features. While it is smooth everywhere, it is written with different expressions for $x > 0$ and $x < 0$. Remarkably, the same phenomenon occurs: the local \star -eigen- \star equations are satisfied for x positive and negative (while the local \star -eigen

⁶This is the terminology used in [11].

equations are not). It seems, therefore, that the \star -eigen- \star equation is generally relevant to the matching of Wigner functions; it can be solved piece-wise and its solutions then matched.

Section 6 is our conclusion.

2 Simple form of the \star -eigen- \star equation

Consider the Hamiltonian

$$H_\alpha = p^2 + e^{2\alpha x} + V(x) , \quad (5)$$

where $V(x)$ is an arbitrary, regular potential. In the $\alpha \rightarrow \infty$ limit, an infinite potential wall is formed at $x = 0$, so that motion is restricted to the negative x -axis.

In [2], it was shown that in the same limit, the stationary Wigner function satisfies the rather cumbersome equation⁷

$$\begin{aligned} & \frac{1}{16} \partial_x^4 \rho(x, p) + \frac{(p^2 + E)}{2} \partial_x^2 \rho(x, p) + (p^4 - 2Ep + E^2) \rho(x, p) \\ & + (p^2 - E) \operatorname{Re} [V(x) \star \rho(x, p)] - p \partial_x \operatorname{Im} [V(x) \star \rho(x, p)] \\ & - \frac{1}{4} \partial_x^2 \operatorname{Re} [V(x) \star \rho(x, p)] - \operatorname{Im} [V(x) \star p \partial_x \rho(x, p)] \\ & + \operatorname{Im} \{V(x) \star \operatorname{Im} [V(x) \star \rho(x, p)]\} + \operatorname{Re} \{V(x) \star \operatorname{Re} [V(x) \star \rho(x, p)]\} \\ & + \operatorname{Re} \left\{ V(x) \star \left[\left(p^2 - E - \frac{1}{4} \partial_x^2 \right) \rho(x, p) \right] \right\} = 0 , \end{aligned} \quad (6)$$

for $x < 0$. Clearly, this equation reduces to (3) when $V = 0$. We will now show that (6) can be simplified substantially, to (4) above.

First, write [3]

$$f \star g = (f, g) + i [f, g] , \quad (7)$$

where

$$(f, g) := f \cos \left\{ \frac{\hbar}{2} (\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x) \right\} g , \quad (8)$$

and

$$[f, g] := f \sin \left\{ \frac{\hbar}{2} (\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x) \right\} g , \quad (9)$$

so that

$$(f, g) = (g, f) , \quad [f, g] = -[g, f] . \quad (10)$$

⁷Henceforth, we will set $\hbar = 1$.

If f and g are both real, then so are (f, g) and $[f, g]$, and we also have

$$\text{Re}(f \star g) = (f, g), \quad \text{Im}(f \star g) = [f, g]. \quad (11)$$

Finally, the following identities

$$\begin{aligned} [[f, g], h] + [[h, f], g] + [[g, h], f] &= 0 \\ [(f, g), h] + ([h, f], g) + ([h, g], f) &= 0 \\ [(f, g), h] + [(h, f), g] + [(g, h), f] &= 0 \end{aligned} \quad (12)$$

ensure the associativity of the star product.

Now, $V = 0$ in (6) must result in the equation (3). Therefore, the first line of (6) can be rewritten as $(p^2 - E) \star \rho \star (p^2 - E)$, as can be verified directly. Also notice that

$$\begin{aligned} -p \partial_x f(x, p) &= \text{Im}(p^2 \star f(x, p)) = [p^2, f(x, p)] \\ (p^2 - \frac{1}{4} \partial_x^2) f(x, p) &= \text{Re}(p^2 \star f(x, p)) = (p^2, f(x, p)), \end{aligned} \quad (13)$$

for real f . Equation (6) can therefore be simplified to

$$\begin{aligned} &(p^2 - E) \star \rho \star (p^2 - E) \\ &+ (p^2 - E, (V, \rho)) + [p^2 - E, [V, \rho]] + [V, [p^2 - E, \rho]] \\ &+ [V, [V, \rho]] + (V, (V, \rho)) + (V, (p^2 - E, \rho)) = 0. \end{aligned} \quad (14)$$

Here we have also used that $[E, f] = 0$ for any f , since E is a constant. It is then a simple matter to verify that (14) reduces to

$$(H - E, (H - E, \rho)) + [H - E, [H - E, \rho]] = 0. \quad (15)$$

Here we have defined the Hamiltonian away from the wall as

$$H = p^2 + V(x). \quad (16)$$

But (15) is just (4), the \star -eigen- \star equation.

It is also interesting to notice that the \star -eigen- \star equation can be written as

$$(H - E) \bar{\star} ((H - E) \star \rho) = 0. \quad (17)$$

Here $\bar{\star} = \exp[-i(\bar{\partial}_x \bar{\partial}_p - \bar{\partial}_p \bar{\partial}_x)/2]$ is the complex conjugate of \star .

3 Time evolution and the \star -eigen- \star equation

So far we have only considered the Wigner functions

$$\rho(x, p) = \frac{1}{\pi} \int_{-\infty}^{\infty} dy e^{-2ipy} \psi(x+y) \bar{\psi}(x-y) \quad (18)$$

derived from the density matrices for a stationary energy eigenstate. These Wigner functions have no explicit dependence on time. We now want to study the time dependence of Wigner functions. Let us denote by $R(x, p; t)$ the symbol of a density matrix element that has explicit time dependence, in order to distinguish it from one with none.

It is the time-independent wave function $\psi(x)$ that enters (18). Simply introducing its time dependence, $\Psi(x, t) := \psi(x) e^{-iEt}$, has no effect. We must therefore consider Wigner functions that are the symbols of off-diagonal density matrix elements in the stationary state basis:

$$\begin{aligned} R_{12}(x, p; t) &= \frac{1}{\pi} \int_{-\infty}^{\infty} dy e^{-2ipy} \Psi_1(x+y, t) \bar{\Psi}_2(x-y, t) \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} dy e^{-2ipy} \psi_1(x+y, t) e^{-iE_1 t} \bar{\psi}_2(x-y) e^{iE_2 t}. \end{aligned} \quad (19)$$

If the wave functions satisfy the Schrödinger equation, then we have

$$\frac{\partial R_{12}}{\partial t} = \frac{1}{i\hbar} (H \star R_{12} - R_{12} \star H) = \frac{1}{i\hbar} [H, R_{12}], \quad (20)$$

the fundamental dynamical equation.

From the pure deformation quantization point of view, (20) must be the starting point. So, how are the \star -eigen equations derived from it? Substituting the ansatz

$$R_{12}(x, p; t) = \rho_{12}(x, p) \exp[-i(E_1 - E_2)t] \quad (21)$$

yields

$$[H, \rho_{12}] = (E_1 - E_2) \rho_{12}. \quad (22)$$

The \star -eigen equations are therefore obtained if

$$(H, \rho_{12}) = (E_1 + E_2) \rho_{12} \quad (23)$$

can also be derived.

In [3], this was done by introducing a complex time $t \rightarrow z := t - is$. Here we will use the same trick to find a dynamical equation that corresponds to the \star -eigen- \star equation (4), or the generalization thereof:

$$[H(x, p) - E_1] \star \rho_{12}(x, p) \star [H(x, p) - E_2] = 0. \quad (24)$$

To that end, we write

$$\tilde{R}_{12}(x, p, z) := \frac{1}{\pi} \int_{-\infty}^{\infty} dy e^{-2ipy} \psi_1(x+y) e^{-iE_1 z} \bar{\psi}_2(x-y) e^{iE_2 \bar{z}}, \quad (25)$$

and find

$$\begin{aligned} H(x, p) \star \tilde{R}_{12}(x, p, z) &= H(x + \frac{i}{2} \vec{\partial}_p, p - \frac{i}{2} \vec{\partial}_x) \tilde{R}_{12}(x, p, z) \\ &= c_{12} \int_{-\infty}^{\infty} dy e^{-2ipy} H(x+y, p - \frac{i}{2} \vec{\partial}_x) \psi_1(x+y) \bar{\psi}_2(x-y) \\ &= c_{12} \int_{-\infty}^{\infty} dy e^{-2ipy} H(x+y, \frac{i}{2} \vec{\partial}_y - \frac{i}{2} \vec{\partial}_x) \psi_1(x+y) \bar{\psi}_2(x-y), \end{aligned} \quad (26)$$

where we have set

$$c_{12} := \exp[-i(E_1 z - E_2 \bar{z})]/\pi \quad (27)$$

to save writing. Integrating by parts, this becomes

$$\begin{aligned} c_{12} \int_{-\infty}^{\infty} dy e^{-2ipy} H(x+y, -\frac{i}{2} \vec{\partial}_x - \frac{i}{2} \vec{\partial}_x) \psi_1(x+y) \bar{\psi}_2(x-y) \\ = E_1 \tilde{R}_{12}(x, p, z). \end{aligned} \quad (28)$$

Similarly, we can show that

$$\tilde{R}_{12}(x, p, z) \star H(x, p) = \tilde{R}_{12}(x, p, z) E_2. \quad (29)$$

Consequently, the dynamical equation

$$\left(i \frac{\partial}{\partial z} - H \right) \star \tilde{R}_{12}(x, p, z) \star \left(-i \frac{\partial}{\partial \bar{z}} - H \right) = 0 \quad (30)$$

is obeyed.

If we substitute

$$\tilde{R}_{12}(x, p, z) = \rho_{12}(x, p) \exp[-i(E_1 z - E_2 \bar{z})], \quad (31)$$

then

$$(E_1 - H) \star \rho_{12} \star (E_2 - H) = 0 \quad (32)$$

follows. As a special case, when $\psi_1 = \psi_2$, the \star -eigen- \star equation (4) is retrieved. These equations can be solved to recover the physical time evolution of the density matrix symbols.

Note that if $z = \bar{z}$, then the \star -eigen- \star equation (4) for the “diagonal” Wigner function could not have been derived in this way. The complexification of time used here is therefore necessary, if somewhat artificial, as well as having appeared before in [3].

4 Potentials with zero-size features: contact interactions

The main point of this article is that the \star -eigen- \star equation applies more generally. In particular, it applies to more general potentials. Included are those with zero-size features, describing the so-called contact interactions, considered in this section.

4.1 Robin boundary conditions on the half line

As our first example, consider an infinite potential wall at $x = 0$ that prevents motion on the positive x -axis. Equivalently, we can consider motion restricted to the half-line $x < 0$, with a point interaction at $x = 0$ that is necessary for a self-adjoint Hamiltonian. The point interaction has the effect of imposing boundary conditions on the wave function.

The most general point interaction at $x = 0$ results in the most general, mixed or Robin, boundary conditions

$$\psi(0) + L\psi'(0) = 0 \quad (33)$$

on the wave function $\psi(x)$. Here $\psi' = d\psi/dx$ and L is a real length scale.

A (scattering) wave function satisfying these boundary conditions is [11]

$$\psi(x) \propto e^{ikx} + e^{i\delta_k} e^{-ikx}, \quad (34)$$

for $x \leq 0$, if

$$kL = \cot(\delta_k/2). \quad (35)$$

Notice that $\delta_{-k} = -\delta_k$. These Robin boundary conditions interpolate between the standard Dirichlet ones at $L = 0$, and the Neumann boundary conditions for $L = \infty$. In these cases, $\delta_k = \pi$ and 0, respectively.

The Wigner function derived from this wave function by a Weyl-Wigner transform is, for $x < 0$,

$$\rho = \frac{\sin[2(p-k)x]}{(p-k)} + \frac{\sin[2(p+k)x]}{(p+k)} + 2 \cos(2kx - \delta_k) \frac{\sin(2px)}{p}. \quad (36)$$

It satisfies $\rho(x, -p) = \rho(x, +p)$, and $\rho(0, p) = 0$.

Clearly, this Wigner function is real, and depends on x . As a consequence, we know it cannot satisfy the \star -eigen equations (1) away from $x = 0$. By (11), the \star -eigen equations imply

$$[p^2, \rho] = p \partial_x \rho = 0, \quad (37)$$

for $\rho = \bar{\rho}$.

In the free case ($V = 0$), we can understand the problem in a slightly deeper way. There is a conflict between the reality of solutions of the \star -eigen equation and the presence of both momenta $\pm\sqrt{E}$, such as is necessary when reflections occur. See the Appendix.

The alternative form of the Wigner function

$$\begin{aligned}\rho = & \sin[2(p - k)x]/(p - k) + \sin[2(p + k)x]/(p + k) \\ & + \sin[2(p - k)x + \delta_k]/p + \sin[2(p + k)x - \delta_k]/p\end{aligned}\quad (38)$$

makes it straightforward to see that the \star -eigen- \star equation is obeyed, however. With $H = p^2$ and $E =: k^2$, it becomes

$$\left[(p^2 - k^2)^2 + 2(p^2 + k^2) \partial_{(2x)}^2 + \partial_{(2x)}^4 \right] \rho = 0. \quad (39)$$

But

$$\begin{aligned}& \left[(p^2 - k^2)^2 + 2(p^2 + k^2) \partial_{(2x)}^2 + \partial_{(2x)}^4 \right] \sin[2(p \pm k)x \mp \delta_k] \\ &= \left[(p^2 - k^2)^2 - 2(p^2 + k^2)(p \pm k)^2 + (p \pm k)^4 \right] \sin[2(p \pm k)x \mp \delta_k] \\ &= (p \pm k)^2 \left[(p \mp k)^2 - 2(p^2 + k^2) + (p \pm k)^2 \right] \sin[2(p \pm k)x \mp \delta_k]\end{aligned}\quad (40)$$

which clearly vanishes. The Wigner function (36) therefore satisfies the \star -eigen- \star equation.

Incidentally, just as $(p^2 - k^2) \star \sin[2(p \pm k)x] \star (p^2 - k^2) = 0$, so does $(p^2 - k^2) \star \cos[2(p \pm k)x] \star (p^2 - k^2) = 0$. Consequently, the four fundamental solutions of the free \star -eigen- \star equation can be taken to be

$$\begin{aligned}& \{ \cos(2kx) \cos(2px), \cos(2kx) \sin(2px), \sin(2kx) \cos(2px), \sin(2kx) \sin(2px) \} \\ & \text{or } \{ \cos[2(p + k)x], \cos[2(p - k)x], \sin[2(p + k)x], \sin[2(p - k)x] \} , \\ & \text{or } \{ e^{2i(p+k)x}, e^{2i(p-k)x}, e^{-2i(p+k)x}, e^{-2i(p-k)x} \} .\end{aligned}\quad (41)$$

The third basis can also be found by realizing that $p^2 - k^2 = (p \pm k) \star (p \mp k)$, and that

$$(p \pm k) \star e^{-2i(p \pm k)x} = 0 \rightarrow e^{2i(p \pm k)x} \star (p \pm k) = 0. \quad (42)$$

Any linear combination of the four solutions from any of these bases, with x -independent coefficient functions, satisfies the \star -eigen- \star equation.

A bound state also exists for $L > 0$, with wave function

$$\psi(x) = \theta(-x) \sqrt{\frac{2}{L}} e^{x/L}, \quad (43)$$

of energy $E = -1/L^2$.

Below we will need to consider the more general situation when $H - E = p^2 + \kappa^2$, with $\kappa^2 > 0$. Now $p^2 + \kappa^2 = (p \pm i\kappa) \star (p \mp i\kappa)$, and

$$(p \pm i\kappa) \star e^{-2ipx} e^{\pm 2\kappa x} = 0 \rightarrow e^{2ipx} e^{\pm 2\kappa x} \star (p \mp i\kappa) = 0. \quad (44)$$

Therefore, any of

$$\left\{ e^{\pm 2ipx} e^{\pm 2\kappa x} \right\} \quad (45)$$

satisfies the \star -eigen- \star equation, as does any linear combination with x -independent coefficient functions.

The Wigner-Weyl transform of the wave function (43) produces the Wigner function

$$\pi \rho[\psi] = -\frac{2}{L} \frac{1}{p} \sin(2px) e^{2x/L}, \quad (46)$$

for $x < 0$, while ρ vanishes for $x > 0$. Identifying $\kappa = 1/L$, then, by (45), the \star -eigen- \star equation is satisfied away from the wall at $x = 0$.

4.2 General point interaction on the real line

A four-parameter family of point interactions exists in one-dimensional quantum mechanics (see [12] and [13], e.g.). The corresponding potentials include the Dirac delta-function, that has the effect of relating the values of the wave function and its derivative on its two sides. The general point interaction imposes more general matching conditions involving the wave function and its x -derivatives on its sides.

With a point interaction at $x = 0$, and wave function

$$\psi(x) = \theta(-x) \psi_-(x) + \theta(x) \psi_+(x), \quad (47)$$

the general matching conditions are

$$\begin{aligned} -\psi'_+(0) - \alpha \psi'_-(0) &= \beta \psi_-(0) \\ -\delta \psi'_-(0) - \gamma \psi_-(0) &= \psi_+(0). \end{aligned} \quad (48)$$

Here ψ' indicates the space derivative of the wave function, and the real parameters $\alpha, \beta, \gamma, \delta$ are related by

$$\alpha\gamma - \beta\delta = 1. \quad (49)$$

For any wave function of the form (47), the Wigner function is

$$\pi \rho[\psi] = \theta(-x) \int_x^{-x} dy e^{-2ipy} \psi_-(x+y) \bar{\psi}_-(x-y)$$

$$\begin{aligned}
& + \int_{-\infty}^{-|x|} dy e^{-2ipy} \psi_-(x+y) \bar{\psi}_+(x-y) \\
& + \int_{|x|}^{\infty} dy e^{-2ipy} \psi_+(x+y) \bar{\psi}_-(x-y) \\
& + \theta(x) \int_{-x}^x dy e^{-2ipy} \psi_+(x+y) \bar{\psi}_+(x-y) . \tag{50}
\end{aligned}$$

Consider first the bound state for a point interaction at $x = 0$, of energy $E = -\kappa^2$, present if

$$\beta + \delta\kappa^2 + \kappa(\alpha + \gamma) = 0, \text{ with } \kappa > 0 . \tag{51}$$

The wave function is

$$\psi_{\pm}(x) = \psi_{\pm}(0) e^{\mp\kappa x} , \tag{52}$$

with

$$\frac{\psi_+(0)}{\psi_-(0)} = \alpha + \frac{\beta}{\kappa} = -\gamma - \delta\kappa . \tag{53}$$

A simple calculation gives

$$\begin{aligned}
\pi \rho[\psi] &= -\frac{e^{2\kappa x} e^{2ipx}}{2i} \bar{\psi}_-(0) \left\{ \frac{\psi_-(0)}{p} - \frac{\psi_+(0)}{p - i\kappa} \right\} \\
&+ \frac{e^{2\kappa x} e^{-2ipx}}{2i} \psi_-(0) \left\{ \frac{\bar{\psi}_-(0)}{p} - \frac{\bar{\psi}_+(0)}{p + i\kappa} \right\} , \tag{54}
\end{aligned}$$

for $x < 0$, and

$$\begin{aligned}
\pi \rho[\psi] &= \frac{e^{-2\kappa x} e^{2ipx}}{2i} \bar{\psi}_+(0) \left\{ \frac{\psi_+(0)}{p} - \frac{\psi_-(0)}{p + i\kappa} \right\} \\
&- \frac{e^{-2\kappa x} e^{-2ipx}}{2i} \psi_+(0) \left\{ \frac{\bar{\psi}_+(0)}{p} - \frac{\bar{\psi}_-(0)}{p - i\kappa} \right\} , \tag{55}
\end{aligned}$$

for $x > 0$. In view of (45), we see that the Wigner function satisfies the \star -eigen- \star equation away from the point interaction.

Let us now consider the Wigner function related to a scattering wave function [13]

$$\psi(x) = \theta(-x) [e^{ikx} + R e^{-ikx}] + \theta(x) T e^{ikx} \tag{56}$$

satisfies the matching conditions (48), with the constants

$$\begin{aligned}
T &= -2ik/D , \quad R = [\beta + \delta k^2 + ik(\alpha - \gamma)]/D , \\
D &= -\beta + \delta k^2 + ik(\alpha + \gamma) . \tag{57}
\end{aligned}$$

The transmission and reflection amplitudes satisfy

$$|R|^2 + |T|^2 = 1 , \tag{58}$$

because of (49). Putting $\psi_-(x) = e^{ikx} + Re^{-ikx}$ and $\psi_+(x) = Te^{ikx}$ in (50) produces

$$\begin{aligned}\pi \rho[\psi](x < 0) = & - \left\{ \frac{\text{Re}(1-T)}{p-k} + \frac{\text{Re}(R)}{p} \right\} \sin[2(p-k)x] \\ & - \left\{ \frac{|R|^2}{p+k} + \frac{\text{Re}[(1-T)\bar{R}]}{p} \right\} \sin[2(p+k)x] \\ & - \left\{ \frac{\text{Im}(1-T)}{p-k} + \frac{\text{Im}(R)}{p} \right\} \cos[2(p-k)x] \\ & - \left\{ \frac{\text{Im}[(1-T)\bar{R}]}{p} \right\} \cos[2(p+k)x],\end{aligned}\quad (59)$$

for $x < 0$, and for $x > 0$:

$$\begin{aligned}\pi \rho[\psi](x > 0) = & - \left\{ \frac{\text{Re}[T(1-\bar{T})]}{p-k} + \frac{\text{Re}(T\bar{R})}{p} \right\} \sin[2(p-k)x] \\ & + \left\{ \frac{\text{Im}(T)}{p-k} + \frac{\text{Im}(T\bar{R})}{p} \right\} \cos[2(p-k)x].\end{aligned}\quad (60)$$

Interestingly, these results can be written in a more compact form as

$$\begin{aligned}-\pi \rho[\psi](x < 0) = & \text{Im} \left\{ \left(\frac{1-T}{p-k} + \frac{R}{p} \right) e^{2i(p-k)x} \right. \\ & \left. + \left(\frac{R\bar{R}}{p+k} + \frac{(1-T)\bar{R}}{p} \right) e^{2i(p+k)x} \right\}\end{aligned}\quad (61)$$

for $x < 0$, and for $x > 0$:

$$-\pi \rho[\psi](x > 0) = \text{Im} \left\{ \left(\frac{(1-T)\bar{T}}{p-k} + \frac{R\bar{T}}{p} \right) e^{2i(p-k)x} \right\}.\quad (62)$$

These expressions make it clear that the Wigner functions do not satisfy the \star -eigen equations, but do obey the \star -eigen- \star equation.

A subtlety must be discussed here, however. When performing the integrations for the “ \pm cross terms”

$$2\text{Re} \int_{|x|}^{\infty} dy e^{-2ipy} \psi_+(x+y) \bar{\psi}_-(x-y)\quad (63)$$

of (50), we have dropped oscillating contributions at $y = \pm\infty$. This could be implemented formally by changing $p \rightarrow p - i\epsilon$, $0 < \epsilon$, in (63), then taking the limit $\epsilon \rightarrow 0$ after integrating over y .

The use of non-normalizable plane waves as the scattering wave functions is the origin of the undefined terms that we omitted. Scattering is certainly treatable with Wigner functions, by using wave packets [14]. Such an analysis is beyond our scope, however, since we are mainly interested in the time-independent Wigner functions and their equations of motion.

4.3 Finite potential jump

If the usefulness of the \star -eigen- \star equation is tied to the zero-size property of a feature of the relevant potential, then it should apply to a finite potential jump, not just to an infinite one. Here we show that this is indeed the case.

The Wigner function for the potential $V(x) = V_0 \theta(x)$, with $E < V_0$, was studied in [15]. The relevant wave function can be expressed as

$$\psi(x) \propto \theta(-x) \cos(kx - \alpha/2) + \theta(x) \cos(\alpha/2) e^{-\kappa x}, \quad (64)$$

where $E = k^2$ again, and $\kappa^2 := V_0 - E$. For continuity of $d\psi/dx$ at $x = 0$,

$$e^{i\alpha} = \frac{ik + \kappa}{ik - \kappa} \quad (65)$$

is required.

For $x < 0$, the corresponding Wigner function is [15]

$$\begin{aligned} & - \frac{k[k(2p - k) + \kappa^2]}{[\kappa^2 + (2p - k)^2]4p(p - k)} \sin[2(p - k)x] \\ & - \frac{k[k(2p + k) - \kappa^2]}{[\kappa^2 + (2p + k)^2]4p(p + k)} \sin[2(p + k)x] \\ & + \frac{\kappa k \cos[2(p - k)x]}{2p[\kappa^2 + (2p - k)^2]} - \frac{\kappa k \cos[2(p + k)x]}{2p[\kappa^2 + (2p + k)^2]}, \end{aligned} \quad (66)$$

up to a multiplicative constant. By (41), we see that it satisfies the \star -eigen- \star equation for $H = p^2$, valid for $x < 0$.

The Wigner function is proportional to

$$k^2 e^{-2\kappa x} \left\{ \frac{4\kappa}{[(2p + k)^2 + \kappa^2][(2p - k)^2 + \kappa^2]} \cos(2px) \right. \\ \left. + \frac{(k^2 + \kappa^2 - 4p^2)}{[(2p + k)^2 + \kappa^2]2p[(2p - k)^2 + \kappa^2]} \sin(2px) \right\}. \quad (67)$$

for $x > 0$, where $H - E = p^2 + \kappa^2$. The \star -eigen- \star equation therefore becomes

$$\left[(p^2 + \kappa^2)^2 + 2(p^2 - \kappa^2) \partial_{(2x)}^2 + \partial_{(2x)}^4 \right] \rho = 0. \quad (68)$$

But

$$\begin{aligned} & \left[(p^2 + \kappa^2)^2 + 2(p^2 - \kappa^2) \partial_{(2x)}^2 + \partial_{(2x)}^4 \right] e^{-2\kappa x} e^{\mp 2ipx} \\ &= (p \mp i\kappa)^2 \left[(p \mp i\kappa)^2 - 2(p^2 - \kappa^2) + (p \pm i\kappa)^2 \right] e^{-2\kappa x} e^{\mp 2ipx} = 0. \end{aligned} \quad (69)$$

We conclude that the Wigner function (67) satisfies the \star -eigen- \star equation for $x > 0$ as well. Clearly, (69) amounts to an explicit confirmation of (45).

For $E > V_0$, we have the extended wave function

$$\psi_-(x) = e^{ikx} + R e^{-ikx}, \quad \psi_+(x) = T e^{i\ell x}, \quad (70)$$

where $k^2 := E$, and $\ell^2 := E - V_0$. As discussed in the last section, we use

$$\begin{aligned} \pi \rho[\psi] &= \theta(-x) \int_x^{-x} dy e^{-2ipy} \psi_-(x+y) \bar{\psi}_-(x-y) \\ &+ \lim_{\epsilon \rightarrow 0^+} 2 \operatorname{Re} \int_{|x|}^{\infty} dy e^{-2i(p-i\epsilon)y} \psi_+(x+y) \bar{\psi}_-(x-y) \\ &+ \theta(x) \int_{-x}^x dy e^{-2ipy} \psi_+(x+y) \bar{\psi}_+(x-y). \end{aligned} \quad (71)$$

For $x > 0$, this gives

$$\pi \rho[\psi] = \operatorname{Im} \left\{ e^{-2ix(p-\ell)} \left[-\frac{|T|^2}{p-\ell} + \frac{2T\bar{R}}{2p+k-\ell} + \frac{2T}{2p-k-\ell} \right] \right\}; \quad (72)$$

and for $x < 0$,

$$\begin{aligned} \pi \rho[\psi] &= \operatorname{Im} \left\{ e^{-2ix(p-k)} \left[\frac{1}{p-k} - \frac{2\bar{T}}{2p-k-\ell} + \frac{\bar{R}}{p} \right] \right. \\ &\quad \left. + e^{-2ix(p+k)} \left[\frac{|R|^2}{p+k} - \frac{2\bar{T}R}{2p+k-\ell} + \frac{R}{p} \right] \right\}. \end{aligned} \quad (73)$$

Putting $\ell = k$ in these formulas yields the results (62) and (61), as should be. Again, it is clear that while these real and x -dependent Wigner functions cannot obey the \star -eigen equations, they do satisfy the \star -eigen- \star equation.

Incidentally, notice that our conclusions are similar for the two cases $E < V_0$ and $E > V_0$. Interestingly, no non-Newtonian (para-classical) reflections occur for $E < 0$, but they do for $E > V_0$.

5 Matching

So far, our results seem to support the hypothesis that it is the zero-size feature of potentials that are necessary for the \star -eigen- \star equation to be useful.

However, consider a potential that can only be written simply in different regions of the x -axis, but is smooth and does not have any physical characteristics at the points separating those regions. As a simple example, we will treat the classical Hamiltonian

$$H = p^2 + \theta(x)x^2, \quad (74)$$

describing a free particle for $x < 0$, and a harmonic force acting when $x > 0$. The potential $V(x) = \theta(x)x^2$ has no special properties at $x = 0$ – it is completely smooth there, for example.

In Schrödinger quantum mechanics, one would just solve the Schrödinger equation locally, i.e., in each region, and then match the solutions. Even contact interactions simply impose matching conditions on the wave functions across the zero-size features of the potential. The systems we have considered so far make it clear that naïve matching does not also work for quantum mechanics in phase space: one cannot obtain the correct Wigner function by solving the \star -eigen equation in different regions, and then matching them. Is the \star -eigen- \star equation still relevant even in the absence of zero-size potential features?

A simple stationary state for this system of Hamiltonian (74), of energy $E = 1$, has a wave function of the form (47) with

$$\psi_-(x) = \cos(x), \quad \text{and} \quad \psi_+(x) = e^{-x^2/2}. \quad (75)$$

The corresponding Wigner function

$$\rho(x, p) = \theta(-x)\rho_-(x, p) + \theta(x)\rho_+(x, p) \quad (76)$$

can be calculated using (50), or (71). We find

$$\begin{aligned} 4\rho_-(x, p) &= \cos[2(p-1)x] \left\{ 2\sqrt{2\pi} e^{-(2p-1)^2/2} \right\} \\ &\quad + \cos[2(p+1)x] \left\{ 2\sqrt{2\pi} e^{-(2p+1)^2/2} \right\} \\ &\quad - \sin[2(p-1)x] \left\{ \frac{1}{p-1} + \frac{1}{p} + 2i\sqrt{2\pi} \operatorname{erf} \left[\frac{i(2p-1)}{\sqrt{2}} \right] e^{-(2p-1)^2/2} \right\} \\ &\quad - \sin[2(p+1)x] \left\{ \frac{1}{p+1} + \frac{1}{p} + 2i\sqrt{2\pi} \operatorname{erf} \left[\frac{i(2p+1)}{\sqrt{2}} \right] e^{-(2p+1)^2/2} \right\}. \end{aligned} \quad (77)$$

This is a real function that depends on x . It therefore does not satisfy (the imaginary parts of) the free \star -eigen equations. Since its x -dependence is described by a linear combination of the functions of (41), however, it does satisfy the free \star -eigen- \star equation.

So, the phenomenon of satisfying the local \star -eigen- \star equation instead of the local \star -eigen equation, has nothing to do with the special features (zero-size features, e.g.) of potentials. Nothing unusual is described by (74) at $x = 0$.

The Wigner function for $x > 0$ is

$$\begin{aligned} 2\sqrt{2\pi}\rho_+(x, p) = & e^{2ix(p+1)-(2p+1)^2/2} \operatorname{erfc}[\sqrt{2}x + i(2p+1)/\sqrt{2}] \\ & + e^{-2ix(p+1)-(2p+1)^2/2} \operatorname{erfc}[\sqrt{2}x - i(2p+1)/\sqrt{2}] \\ & - e^{2ix(p-1)-(2p-1)^2/2} \operatorname{erfc}[\sqrt{2}x + i(2p-1)/\sqrt{2}] \\ & - e^{-2ix(p-1)-(2p-1)^2/2} \operatorname{erfc}[\sqrt{2}x - i(2p-1)/\sqrt{2}] \\ & + \sqrt{2}e^{-x^2-p^2} \{ \operatorname{erf}(x+ip) + \operatorname{erf}(x-ip) \}. \end{aligned} \quad (78)$$

For the local Hamiltonian, $H_+ = p^2 + x^2$, the imaginary parts of the \star -eigen equations require

$$(p\partial_x - x\partial_p)\rho(x, p) = 0, \quad (79)$$

implying that the solution should be a function only of H_+ , not of both x and p , independently. Clearly, then, (78) does not satisfy the local \star -eigen equation.

With $H \rightarrow H_+ = x^2 + p^2$, however, the local ($x > 0$) \star -eigen- \star equation is

$$\begin{aligned} 0 = & [(x^2 + p^2 - E)^2 - 1]\rho - 2x\partial_x\rho - 2p\partial_p\rho \\ & - (x^2 + p^2 - E)(\partial_x^2 + \partial_p^2)\rho/2 + x^2\partial_p^2\rho - 2xp\partial_x\partial_p\rho + p^2\partial_x^2\rho \\ & + (\partial_x^2 + \partial_p^2)^2\rho/16. \end{aligned} \quad (80)$$

With $E = 1$, the local Wigner function (78) satisfies this equation.

These results are consistent with the following conjecture. Suppose

$$H = \theta(-x)H_- + \theta(x)H_+, \quad (81)$$

so that the Wigner function has the form (76). Then, although $(H_\pm - E) \star \rho_\pm$ does not necessarily vanish, we still have

$$(H_\pm - E) \star \rho(x, p) \star (H_\pm - E) = 0. \quad (82)$$

6 Conclusion

In [2], an alternative approach to the pure deformation quantization of systems with infinite potential walls was derived from first principles. Here we have demonstrated that its substitute equation, the \star -eigen- \star equation, is much simpler than was previously thought, taking the form (4) for potentials constructed from infinite walls/wells, with an arbitrary, but regular, additional potential.

Not only is the \star -eigen- \star equation simple, it is also more generally applicable than was realized. For one thing, we showed here it can be modified to describe the time evolution of such systems (see section 3). More importantly, we demonstrated its applicability to other contact interactions, i.e. to other potentials with zero-size features. The one-dimensional systems we treated here are: infinite walls with Robin boundary conditions, general point interactions, and, most significantly, finite potential jumps. For all these cases, the local \star -eigen equations are not satisfied, while the local \star -eigen- \star equation is.

Most revealingly, the same properties were shown to apply to system with a potential written differently for two adjacent regions of configuration space, as in (74). Specifically, the particle experiences different local Hamiltonians for $x < 0$ and $x > 0$, and while the corresponding local \star -eigen equation is not obeyed, the local \star -eigen- \star equation is.

What is common to the final example and the contact interactions is that the Hamiltonian (or potential) is most easily viewed piece-wise, using local expression in each region. In operator quantum mechanics, the wave functions would be found in each region, i.e. locally, and then matched at the boundaries separating the regions.

It seems, therefore, that the \star -eigen- \star equation is generally relevant to the matching of Wigner functions. We conjecture that it can be solved locally, i.e. piece-wise, and its solutions then matched to get the “global”, physical Wigner function.

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Appendix: Free \star -eigen- \star equation

For the free Hamiltonian $H = p^2$, putting $E := k^2$,

$$H - E = (p - k) \star (p + k) = (p + k) \star (p - k) . \quad (83)$$

So (using the associativity of the \star -product) $f_{\pm k}(x, p) := \exp[-2ix(p \mp k)]$ satisfy the \star -eigen- \star equation, since

$$(p \mp k) \star f_{\pm k} = 0 = \bar{f}_{\pm k} \star (p \mp k) . \quad (84)$$

If reflections occur, then both components f_k and f_{-k} must be present.

Equations (84) imply that any linear combination of f_k and f_{-k} satisfies the \star -eigen equation,

$$\begin{aligned} & (p^2 - k^2) \star (\alpha_k f_k + \alpha_{-k} f_{-k}) \\ &= \alpha_k (p + k) \star (p - k) \star f_k + \alpha_{-k} (p - k) \star (p + k) \star f_{-k} = 0 , \end{aligned} \quad (85)$$

and by complex conjugation,

$$\begin{aligned} & (\bar{\alpha}_k \bar{f}_k + \bar{\alpha}_{-k} \bar{f}_{-k}) \star (p^2 - k^2) \\ &= \bar{\alpha}_k \bar{f}_k \star (p - k) \star (p + k) + \bar{\alpha}_{-k} \bar{f}_{-k} \star (p + k) \star (p - k) = 0 . \end{aligned} \quad (86)$$

These solutions of the \star -eigen equation are not real, however. The real combinations $\text{Re}f_k = (f_k + \bar{f}_k)/2$ and $\text{Im}f_k = (f_k - \bar{f}_k)/(2i)$ do not satisfy either of the \star -eigen equations, but do satisfy the \star -eigen- \star equation:

$$\begin{aligned} 0 &= (p^2 - k^2) \star (f_k \pm \bar{f}_k) \star (p^2 - k^2) \\ &= ((p^2 - k^2) \star f_k) \star (p^2 - k^2) \pm (p^2 - k^2) \star (\bar{f}_k \star (p^2 - k^2)) . \end{aligned} \quad (87)$$

This \star -eigen- \star equation is a fourth-order differential equation, whereas the \star -eigen equations are second order. The real combinations $(f_k + \bar{f}_k)/2$ and $(f_k - \bar{f}_k)/(2i)$ do obey a second order equation, of course:

$$0 = (p - k) \star \rho \star (p - k) , \quad (88)$$

but $(f_{-k} + \bar{f}_{-k})/2$ and $(f_{-k} - \bar{f}_{-k})/(2i)$ satisfy a different one:

$$0 = (p + k) \star \rho \star (p + k) . \quad (89)$$

So, there is a conflict between reflections and the reality of Wigner functions if the free \star -eigen equations are to be used. The \star -eigen- \star equation avoids this problem.